

Compactification along lightlike lattices

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Spacetimes obtained by dimensional reduction along lattices containing a lightlike direction can admit semigroup extensions of their isometry groups. We show by concrete examples that such a semigroup can exhibit a natural order, which in turn implies the existence of preferred coordinate charts on the underlying space. Specifically, for spacetimes which are products of an external Minkowski space with an internal two-dimensional Lorentzian space, where one of the lightlike directions has a compact size, the preferred charts consist of "infinite-momentum" frames on the internal space. This implies that fields viewed from this preferred frame acquire extreme values; in particular, some of the off-diagonal components of the higher-dimensional metric, which may be regarded as gauge potentials for a field theory on the external Minkowski factor, vanish. This raises the possibility of regarding known gauge theories as part of more extended field multiplets which have been reduced in size since they are perceived from within an extreme frame. In the case of an external 4-dimensional Minkowski spacetime times a two-dimensional Lorentzian cylinder, the field content as seen in the preferred frame is that of a five-dimensional Kaluza-Klein theory, where the electrodynamic potentials A_μ may depend, in addition to the external spacetime coordinates, on a fifth coordinate along a lightlike direction. The fact that the metric along this direction is zero obstructs the generation of field equations from the Ricci tensor of the overall metric.

I. INTRODUCTION

Spacetimes which are obtained by dimensional reduction along lightlike directions can exhibit peculiarities in their geometrical properties which are not found in other compactifications. The original motivation for the work presented in this paper has been to examine the structure of the isometry group of flat compactified spacetimes when dimensional reduction is performed over a lattice containing a lightlike direction. In the course of our investigation we have uncovered some interesting results about the geometry of higher-dimensional spacetimes M which are product manifolds with an "internal factor" which contains a lightlike circle. It turns out that such spaces can admit semigroup extensions $eI(M)$ of their isometry group $I(M)$; that is to say, they admit smooth global maps $\Lambda : M \rightarrow M$ which are surjective, but no longer injective on M , and which locally preserve the metric, so that they still qualify as "isometries", albeit in a wider sense. Any two such transformations g and g' again combine to a, generally non-invertible, metric-preserving transformation gg' ; but the non-injectiveness implies that global inverses of such maps do not generally exist. This fact accounts for the semigroup structure of these maps.

Such a semigroup extension suggests interesting consequences for the existence of preferred coordinate charts on the space underlying; for, a semigroup of isometries can be given a natural ordering on account of the fact that the product $gg' = h$ of two semigroup elements g and g' still belongs to $eI(M)$, but there may be no way to resolve this product for either g or g' . Below we shall walk through the simplest example possible, namely a cylindrical internal spacetime where the circle of the cylinder is lightlike; we shall see that in this case the preferred coordinate charts consist of "infinite-momentum" frames, i.e., coordinate frames which are related to any other frame by the limit of a series of internal Lorentz transformations, i.e., an infinite boost. Viewed from such a frame, fields defined on the spacetime acquire extreme values; in particular, some of the off-diagonal components of the higher-dimensional metric, which may be regarded as gauge potentials for a field theory on the $(3+1)$ -dimensional external spacetime factor, vanish. This raises the possibility of regarding known gauge theories, with a given number of gauge potentials, as part of a more extended field multiplet, which has been "reduced" in size since it is perceived from within an "extreme" frame. What is more, the semigroup structure implies that all fields defined on the internal spacetime factor must be independent of one of the lightlike coordinates on this factor, namely the one which has acquired a compact size in the process of dimensional reduction. We then end up being able to explain quite naturally how one dimension on the internal manifold must cease to reveal its presence, since all fields must be independent of this dimension.

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Furthermore, the geometry of the lightlike cylinders studied below has another interesting property: This property is given by the fact that all lightlike cylinders, independent of the size of the "compactification radius" along the lightlike direction, are *isometric*. This implies that any physical theory which is formulated covariantly in terms of geometric objects on the higher-dimensional spacetime is automatically independent of the compactification radius along one of the lightlike directions. This then means that the question whether this radius is microscopic or large becomes irrelevant, since no covariant theory can distinguish between two different radii!

Below we shall work out explicitly the example of a 6-dimensional spacetime which is a product of a 4-dimensional Minkowski spacetime and an internal lightlike cylinder. In this case the field content as seen in the "infinite-momentum frame" on the internal space is that of a 5-dimensional Kaluza-Klein theory, where the electrodynamic potentials A_μ may depend, in addition to the four external spacetime coordinates, on a fifth coordinate along a lightlike direction. We discuss some of the problems associated with obtaining meaningful equations of motion for these potentials, which formally are equations on a manifold with *degenerate metric*. This leads outside the established framework of semi-Riemannian geometry, and hence it is not clear how to produce dynamical equations for the potentials A_μ .

Lightlike compactification, or "discretization", has been utilized within the framework of Discrete Light-Cone Quantization, a scheme which has been proposed for gauge field quantization [1] and a lattice approach to string theory [2]; it goes back to Dirac's original idea of light-cone quantization [3] which was later rediscovered by Weinberg [4]. While in Discrete Light-Cone Quantization lightlike directions are compactified (usually one or two), it is assumed that these directions are obtained from a coordinate transformation onto an infinite-momentum reference frame living in ordinary 3+1-dimensional Minkowski spacetime. In contrast, our work focuses on the compactification of lightlike directions in a higher-dimensional covering spacetime. Geometrical aspects of dimensional reduction along lightlike Killing vectors, and action principles in such spaces, were investigated in [5]. These authors studied compactifications along continuous orbits; in contrast, in this work we are dealing with orbits which result from the action of discrete groups.

The plan of the paper is as follows: In section II we motivate the emergence of semigroup extensions from a more abstract point of view by considering how isometry groups of orbit spaces are obtained from isometry groups of associated covering spaces. In section III we discuss a condition that obstructs the existence of semigroup extensions for orbit spaces which are obtained from the action of primitive lattice translations of a suitable lattice on a flat pseudo-Euclidean covering space. In section IV we study in detail the example of two-dimensional Lorentzian cylindrical spaces which are obtained by compactifying a lightlike direction in a Lorentzian covering space. We show that the associated semigroup extension of the isometry group of the cylinder exhibits a natural ordering, which may be interpreted as pointing out preferred, "infinite-momentum", frames on this space. The consequences of these semigroup transformations for scalars, vectors and covectors on the cylinder are examined in sections V and VI. In section VII we regard the cylindrical space as the "internal" factor in a product spacetime whose external factor is Minkowski. We show that the preferred charts on the total space reduce the associated, originally (4+2)-dimensional, Kaluza-Klein theory to an effective (3+1+0)-dimensional Kaluza-Klein scenario, where in the latter case the internal space appears one-dimensional and carries a zero metric. In VIII we summarize our results.

II. ORBIT SPACES AND NORMALIZING SETS

We have mentioned in the introduction that semigroup extensions of isometry groups emerge naturally when dimensional reduction along lightlike directions is performed. In order to see the key point at which such extensions present themselves it is best to approach the subject in a more general way, by motivating the idea of the *extended normalizer* of a set of symmetry transformations. To this end we start by recalling how symmetry groups, and possible extensions thereof, of identification spaces are obtained from the symmetry groups of associated covering spaces. The relevant idea here is that of diffeomorphisms on a covering space which *descend to a quotient space*:

Let M be a connected pseudo-Riemannian manifold with a metric η , and let $I(M)$ be the group of isometries of M . Assume that a discrete subgroup $\Gamma \subset I(M)$ acts properly discontinuously and freely on M ; in this case the natural projection $p : M \rightarrow M/\Gamma$ from the space of orbits, M/Γ , can be made into a covering map, and M becomes a covering space of M/Γ (e.g. [6–9]). In fact there is a unique way to make the quotient M/Γ a pseudo-Riemannian manifold (e.g. [10–14]): In this construction one stipulates that the projection p be a local isometry, which determines the metric on M/Γ . In such a case, we call $p : M \rightarrow M/\Gamma$ a pseudo-Riemannian covering.

In any case the quotient $p : M \rightarrow M/\Gamma$ can be regarded as a fibre bundle with bundle space M , base M/Γ , and Γ as structure group, the fibre over $m \in M/\Gamma$ being the orbit of any element $x \in p^{-1}(m)$ under Γ , i.e. $p^{-1}(m) = \Gamma x = \{\gamma x \mid \gamma \in \Gamma\}$. If $g \in I(M)$ is an isometry of M , then g gives rise to a well-defined map $g_\# : M/\Gamma \rightarrow M/\Gamma$, defined by

$$g_\#(\Gamma x) \equiv \Gamma(gx) \quad , \quad (1)$$

on the quotient space **only** when g preserves all fibres, i.e. when $g(\Gamma x) \subset \Gamma(gx)$ for all $x \in M$. This is equivalent to

saying that $g\Gamma g^{-1} \subset \Gamma$. If this relation is replaced by the stronger condition $g\Gamma g^{-1} = \Gamma$, then g is an element of the *normalizer* $N(\Gamma)$ of Γ in $I(M)$, where

$$N(\Gamma) = \{g \in I(M) \mid g\Gamma g^{-1} = \Gamma\} \quad . \quad (2)$$

The normalizer is a group by construction. It contains all fibre-preserving elements g of $I(M)$ such that g^{-1} is fibre-preserving as well. In particular, it contains the group Γ , which acts trivially on the quotient space; this means that for any $\gamma \in \Gamma$, the induced map $\gamma_{\#} : M/\Gamma \rightarrow M/\Gamma$ is the identity on M/Γ . This follows, since the action of $\gamma_{\#}$ on the orbit Γx , say, is defined to be $\gamma_{\#}(\Gamma x) \equiv \Gamma(\gamma x) = \Gamma x$, where the last equality holds, since Γ is a group.

In this work we are interested in relaxing the equality in the condition defining $N(\Gamma)$; to this end we introduce what we wish to call the *extended normalizer*, denoted by $eN(\Gamma)$, through

$$eN(\Gamma) \equiv \{g \in I(M) \mid g\Gamma g^{-1} \subset \Gamma\} \quad . \quad (3)$$

The elements $g \in I(M)$ which give rise to well-defined maps $g_{\#}$ on M/Γ are therefore precisely the elements of the extended normalizer $eN(\Gamma)$, as we have seen in the discussion above. Such elements g are said to *descend to the quotient space* M/Γ . Hence $eN(\Gamma)$ contains all isometries of M that descend to the quotient space M/Γ ; the normalizer $N(\Gamma)$, on the other hand, contains all those g for which g^{-1} descends to the quotient as well. Thus, $N(\Gamma)$ is the group of all g which descend to **invertible** maps $g_{\#}$ on the quotient space. In fact, the normalizer $N(\Gamma)$ contains all isometries of the quotient space, the only point being that the action of $N(\Gamma)$ is not effective, since $\Gamma \subset N(\Gamma)$ acts trivially on M/Γ . However, Γ is a normal subgroup of $N(\Gamma)$ by construction, so that the quotient $N(\Gamma)/\Gamma$ is a group again, which is now seen to act effectively on M/Γ , and the isometries of M/Γ which descend from isometries of M are in a 1–1 relation to elements of this group. Thus, denoting the isometry group of the quotient space M/Γ as $I(M/\Gamma)$, we have the well-known result that

$$I(M/\Gamma) = N(\Gamma)/\Gamma \quad . \quad (4)$$

Now we turn to the extended normalizer. For an element $g \in eN(\Gamma)$, but $g \notin N(\Gamma)$, the induced map $g_{\#}$ is no longer injective on M/Γ : To see this we observe that now the inclusion in definition (3) is proper, i.e. $g\Gamma g^{-1} \subsetneq \Gamma$. It follows that a $\gamma' \in \Gamma$ exists for which

$$g\gamma g^{-1} \neq \gamma' \quad \text{for all } \gamma \in \Gamma \quad . \quad (5)$$

Take an arbitrary $x \in M$; we claim that

$$g\Gamma g^{-1}x \subsetneq \Gamma x \quad . \quad (6)$$

To see this, assume to the contrary that the sets $g\Gamma g^{-1}x$ and Γx coincide; then a $\gamma_1 \in \Gamma$ exists for which $g\gamma_1 g^{-1}x = \gamma'x$; since g is an element of the extended normalizer $eN(\Gamma)$, $g\gamma_1 g^{-1} \in \Gamma$, i.e., $g\gamma_1 g^{-1} = \gamma_2$, say. It follows that $\gamma_2 x = \gamma'x$ or $\gamma_2^{-1}\gamma'x = x$. The element $\gamma_2^{-1}\gamma'$ belongs to Γ , which, by assumption, acts freely. Free action implies that if a group element has a fixed point then it must be the unit element, implying that $\gamma' = \gamma_2 = g\gamma_1 g^{-1}$, which contradicts (5); therefore, (6) must hold. Eq. (6) can be expressed by saying that the orbit of x is the image of the orbit of $g^{-1}x$ under the action of the induced map $g_{\#}$, $g_{\#}(\Gamma g^{-1}x) = \Gamma x$; but that the $g_{\#}$ -image of the orbit $\Gamma g^{-1}x$, regarded as a set, is properly contained in the orbit of x . The latter statement means that a $\gamma' \in \Gamma$ exists, as in (5), such that $\gamma'x \neq g\gamma g^{-1}x$ for all γ . It follows that $\gamma g^{-1}x \neq g^{-1}\gamma'x$ for all γ , implying that the point $g^{-1}\gamma'x$ is not contained in the orbit of the point $g^{-1}x$. Its own orbit, $\Gamma g^{-1}\gamma'x$, is therefore distinct from the orbit $\Gamma g^{-1}x$ of the point $g^{-1}x$, since two orbits either coincide or are disjoint otherwise. However, the induced map $g_{\#}$ maps $\Gamma g^{-1}\gamma'x$ into

$$g_{\#}(\Gamma g^{-1}\gamma'x) = \Gamma(gg^{-1}\gamma'x) = \Gamma x \quad , \quad (7)$$

from which it follows that $g_{\#}$ maps two distinct orbits into the same orbit Γx , expressing the fact that $g_{\#}$ is not injective. In particular, if g was an isometry of M , then $g_{\#}$ can no longer be a global isometry on the quotient space, since it is not invertible **on the quotient space**. From this fact, or directly from its definition (3), we infer that the extended normalizer is a semigroup, since it contains the identity, and with any two elements g and g' also their product gg' . On the other hand, we shall see below that there are cases where the elements g of the extended normalizer are still locally injective, in particular, injective on the tangent spaces of the compactification; and moreover, they preserve the metric on the tangent spaces, so that they should be regarded as a kind of "generalized isometries".

III. IDENTIFICATIONS OVER LATTICES

So far we have been entirely general with regard to the manifolds M . We now make more specific assumptions: Our covering spaces M are taken to be a flat pseudo-Euclidean space \mathbb{R}_t^n , i.e., \mathbb{R}^n endowed with a pseudo-Euclidean metric η with signature $(-t, +s)$, $t + s = n$. The isometry group $I(\mathbb{R}_t^n)$ of this space is a semidirect product $\mathbb{R}^n \odot O(t, n-1)$, where \mathbb{R}^n denotes the translational factor, and will be denoted by $\mathbb{E}(t, n-t)$. The identification group Γ will be taken as the set of primitive lattice transformations of a lattice in \mathbb{R}_t^n . If the \mathbb{R} -linear span of the lattice vectors has real dimension m , say, the resulting identification space M/Γ is homeomorphic to a product manifold $\mathbb{R}^{n-m} \times T^m$, where T^m denotes an m -dimensional torus. This space inherits the metric from the covering manifold \mathbb{R}_t^n , since the metric is a local object, but the identification changes only the global topology. Thus M/Γ is again a manifold, but may cease to be semi-Riemannian, since the metric on the torus may turn out to be degenerate. Whereas the isometry group of the covering space \mathbb{R}_t^n is $\mathbb{E}(t, n-t)$, the isometry group of the "compactified" space M/Γ is obtained from the normalizer $N(\Gamma)$ of the group Γ in $\mathbb{E}(t, n-t)$ according to formula (4).

For our purposes it is sufficient to consider lattices that contain the origin $0 \in \mathbb{R}_t^n$ as a lattice point. Let $1 \leq m \leq n$, let $\underline{u} \equiv (u_1, \dots, u_m)$ be a set of m linearly independent vectors in \mathbb{R}_t^n ; then the \mathbb{Z} -linear span of \underline{u} ,

$$\text{lat} \equiv \left\{ \sum_{i=1}^m z_i \cdot u_i \mid z_i \in \mathbb{Z} \right\} \quad , \quad (8)$$

is the set of *lattice points* with respect to \underline{u} . Let U denote the \mathbb{R} -linear span of \underline{u} , or equivalently, of lat .

The subset $T_{\text{lat}} \subset \mathbb{E}(t, n-t)$ is the subgroup of all translations in $\mathbb{E}(t, n-t)$ through elements of lat ,

$$T_{\text{lat}} = \{ (t_z, 1) \in \mathbb{E}(t, n-t) \mid t_z \in \text{lat} \} \quad . \quad (9)$$

Elements $(t_z, 1)$ of T_{lat} are called *primitive lattice translations*. T_{lat} is taken as the discrete group Γ of identification maps, which gives rise to the identification space $p : \mathbb{R}_t^n \rightarrow \mathbb{R}_t^n / T_{\text{lat}}$.

We now examine the normalizer and extended normalizer of the identification group: For an element $(t, R) \in \mathbb{E}(t, n-t)$ to be in the extended normalizer $eN(T_{\text{lat}})$ of T_{lat} , the condition

$$(t, R)(t_z, 1)(t, R)^{-1} = (Rt_z, 1) \in T_{\text{lat}} \quad (10)$$

must be satisfied for lattice vectors t_z . In other words, $Rt_z \in \text{lat}$, which means that the pseudo-orthogonal transformation R must preserve the lattice lat , $R\text{lat} \subset \text{lat}$. For an element (t, R) to be in the normalizer, $(t, R)^{-1}$ must be in the normalizer as well, implying $R^{-1}\text{lat} \subset \text{lat}$, so altogether $R\text{lat} = \text{lat}$. The elements R occurring in the normalizer therefore naturally form a subgroup G_{lat} of $O(t, n-t)$; on the other hand, the elements R occurring in the extended normalizer form a *semigroup* $eG_{\text{lat}} \supset G_{\text{lat}}$. Furthermore, no condition on the translations t in (t, R) arises, hence all translations occur in the normalizer as well as in the extended normalizer. Thus, the [extended] normalizer has the structure of a semi-direct [semi-]group

$$N(\Gamma) = \mathbb{R}^n \odot G_{\text{lat}} \quad , \quad (11a)$$

$$eN(\Gamma) = \mathbb{R}^n \odot eG_{\text{lat}} \quad , \quad (11b)$$

where \mathbb{R}^n refers to the subgroup of all translations in $\mathbb{E}(t, n-t)$.

A sufficient condition under which eG_{lat} coincides with G_{lat} is easily found:

Theorem A (Condition). *If the restriction $\eta|_U$ of the metric η to the real linear span U of the lattice vectors is positive- or negative-definite, then $eG_{\text{lat}} = G_{\text{lat}}$.*

Proof:

We start with the case that $\eta|_U$ is positive definite. Assume that $R \in eG_{\text{lat}} \subset O(t, n-t)$, but $R \notin G_{\text{lat}}$. Then R preserves the lattice but is not surjective, i.e., $R\text{lat} \subsetneq \text{lat}$. This implies the series of inclusions

$$\dots R^3\text{lat} \subsetneq R^2\text{lat} \subsetneq R\text{lat} \subsetneq \text{lat} \quad . \quad (12)$$

Now choose an $x \in \text{lat} \setminus R\text{lat}$; then it follows that $Rx \in R\text{lat} \setminus R^2\text{lat}, \dots, R^k x \in R^k\text{lat} \setminus R^{k+1}\text{lat}$. The series of proper inclusions (12) then implies that the elements in the series $k \mapsto R^k x$ are all distinct from each other; furthermore, they are all lattice points, since R preserves the lattice. We then see that the set of lattice points $\{R^k x \mid k \in \mathbb{N}_0\}$ must be infinite. This means that there are elements in this set whose Euclidean norm $\sqrt{(\eta|_U)(R^k x, R^k x)}$ exceeds

every finite bound. However, the transformations R are taken from the overall metric-preserving group $O(t, n-t)$ and hence must also preserve the restricted metric $\eta|U$. The latter statement implies that

$$\cdots (\eta|U)(R^3x, R^3x) = (\eta|U)(R^2x, R^2x) = \eta(Rx, Rx) = \eta(x, x) \quad , \quad (13)$$

hence all elements R^kx have the same norm according to (13). This contradicts the previous conclusion that there are elements for which the norm exceeds all bounds. This shows that our initial assumption $R \notin G_{\text{lat}}$ was wrong.

If $\eta|U$ is assumed to be negative-definite, the argument given above clearly still applies, since the norm of an element x in this case is just given by $\sqrt{-(\eta|U)(x, x)}$. This completes our proof. \blacksquare

We see that the structure of the proof relies on the fact that the restricted metric $\eta|U$ was Euclidean. If this metric were pseudo-Euclidean instead, the possibility that $eG_{\text{lat}} \supsetneq G_{\text{lat}}$ arises. We shall now study an explicit example of this situation.

IV. TWO-DIMENSIONAL CYLINDERS WITH A COMPACT LIGHTLIKE DIRECTION

We now study a specific example of a space, or rather, a family of spaces, for which natural semigroup extensions can be derived from the isometry group of a simply-connected covering space.

The quotient spaces under consideration are two-dimensional cylinders and will be denoted by C_r^2 ; they are defined as follows:

$$C_r^2 \equiv [0, r) \times \mathbb{R} \quad , \quad r \sim 0 \quad , \quad (14)$$

i.e., $[0, r)$ is a circle with perimeter $r > 0$, and we have the identification $0 \sim r$. These spaces shall have canonical coordinates (x^+, x^-) , where $0 \leq x^+ < r$ and $x^- \in \mathbb{R}$. To the coordinates (x^+, x^-) we shall also refer as *lightlike*, for reasons which will become clear shortly. Each of the cylinders C_r^2 is endowed with a metric h which, in lightlike coordinates, is given as

$$h = -dx^+ \otimes dx^- - dx^- \otimes dx^+ \quad . \quad (15)$$

The manifolds C_r^2 have a remarkable property: They are all isometric, the isometry $\phi_{rr'} : C_r^2 \rightarrow C_{r'}^2$ being given by

$$\phi_{rr'}(x^+, x^-) = (x'^+, x'^-) \equiv \left(\frac{r'}{r} x^+, \frac{r}{r'} x^- \right) \quad , \quad (16)$$

where (x^+, x^-) , (x'^+, x'^-) are lightlike coordinates on C_r^2 and $C_{r'}^2$, respectively. Since the metric on $C_{r'}^2$ is given by

$$h' = -dx'^+ \otimes dx'^- - dx'^- \otimes dx'^+ \quad , \quad (17)$$

we see immediately that h and h' are related by pull-back,

$$\phi_{rr'}^* h' = h \quad . \quad (18)$$

This means that the geometry of C_r^2 , and subsequently, any physics built upon covariant geometrical objects on C_r^2 , must be completely insensitive to the size of the "compactification radius" $r/2\pi$. This may suggest that it makes sense to consider the limit $r \rightarrow 0$.

We now show how the cylinders C_r^2 are obtained from a simply-connected two-dimensional covering space: The uncompactified covering space is taken to be \mathbb{R}_1^2 , which is homeomorphic to \mathbb{R}^2 with canonical coordinates $X \equiv (x^0, x^1)$ with respect to the canonical basis $(\mathbf{e}_0, \mathbf{e}_1)$, while the metric in this basis is $\text{diag}(-1, 1)$. The isometry group is $\mathbb{E}(1, 1) = \mathbb{R}^2 \odot O(1, 1)$, where \mathbb{R}^2 acts as translational subgroup $(t, 1)$ according to

$$\mathbb{R}^2 \ni t = \begin{pmatrix} t^0 \\ t^1 \end{pmatrix} \quad , \quad (t, 1) \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} x^0 + t^0 \\ x^1 + t^1 \end{pmatrix} \quad . \quad (19)$$

A faithful 3-dimensional real matrix representation of $\mathbb{E}(1, 1)$ is given by

$$(t, \Lambda) = \begin{pmatrix} \Lambda & t \\ 0 & 1 \end{pmatrix} \quad , \quad (20)$$

satisfying the Poincaré group law $(t, \Lambda)(t', \Lambda') = (\Lambda t' + t, \Lambda \Lambda')$. If Λ lies in the identity component of $O(1, 1)$ then

$$\Lambda = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix} \quad , \quad \alpha \in \mathbb{R} \quad . \quad (21)$$

This representation acts on the coordinates $X \in \mathbb{R}_1^2$ according to

$$(t, \Lambda) X = \begin{pmatrix} \Lambda & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda X + t \\ 1 \end{pmatrix} \quad . \quad (22)$$

We now introduce a one-dimensional lattice with primitive lattice vector $r \mathbf{e}_+$, where $\mathbf{e}_+ \equiv \frac{1}{\sqrt{2}}(\mathbf{e}_0 + \mathbf{e}_1)$, so that the set \underline{u} as defined in section III contains just one element,

$$\underline{u} = \{r \mathbf{e}_+\} \quad . \quad (23)$$

We can introduce lightlike coordinates

$$\begin{pmatrix} x^+ \\ x^- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = M \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \quad , \quad M = M^T = M^{-1} \quad , \quad (24)$$

on the covering space \mathbb{R}_1^2 , so that \mathbf{e}_+ is the basis vector in the x^+ -direction. The set lat as defined in (8) is now given as

$$\text{lat} = \{\mathbf{0}, \pm r \mathbf{e}_+, \pm 2r \mathbf{e}_+, \dots\} \quad . \quad (25)$$

The subset T_{lat} as defined in (9) is now

$$T_{\text{lat}} = \left\{ (k r \mathbf{e}_+, 1) \in \mathbb{E}(1, 1) \mid k \in \mathbb{Z} \right\} \quad . \quad (26)$$

The elements of T_{lat} are the primitive lattice translations, and the identification group is $\Gamma = T_{\text{lat}}$. After taking the quotient of \mathbb{R}_1^2 over (26), we obtain the two-dimensional cylinder C_r^2 as defined in (14).

According to (10), a general Poincare transformation (t, Λ) lies in the extended normalizer of T_{lat} if and only if Λ preserves the lattice (25), which is equivalent to the condition that

$$\Lambda \mathbf{e}_+ = k \cdot \mathbf{e}_+ \quad \text{for some integer } k \quad . \quad (27)$$

To facilitate computations we now transform everything into lightlike coordinates (24). Then the metric $h = \text{diag}(-1, 1)$ takes the form

$$h = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad , \quad (28)$$

while a Lorentz transformation (21) in this basis looks like

$$\Lambda = \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix} \quad . \quad (29)$$

If we allow Λ to take values in the full group $O(1, 1)$, rather than the identity component, we obtain matrices

$$\Lambda = \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix} \quad , \quad -\begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & e^{-\alpha} \\ e^\alpha & 0 \end{pmatrix} \quad , \quad -\begin{pmatrix} 0 & e^{-\alpha} \\ e^\alpha & 0 \end{pmatrix} \quad . \quad (30)$$

Here we have introduced the parity transformation \mathcal{P} , defined by $\mathcal{P}(x^0, x^1) = (x^0, -x^1)$, which acts on lightlike coordinates according to

$$\mathcal{P} \begin{pmatrix} x^+ \\ x^- \end{pmatrix} = \begin{pmatrix} x'^+ \\ x'^- \end{pmatrix} = \begin{pmatrix} x^- \\ x^+ \end{pmatrix} \quad , \quad (31)$$

and the time-reversal transformation $\mathcal{T}(x^0, x^1) = (-x^0, x^1) = -\mathcal{P}(x^0, x^1)$, which acts on lightlike coordinates as

$$\mathcal{T} \begin{pmatrix} x^+ \\ x^- \end{pmatrix} = \begin{pmatrix} x'^+ \\ x'^- \end{pmatrix} = \begin{pmatrix} -x^- \\ -x^+ \end{pmatrix} \quad . \quad (32)$$

Parity and time-reversal are the discrete isometries of the metric (28). The sequence of matrices in (30) can then be written as

$$\Lambda \quad , \quad \mathcal{P} \Lambda = \Lambda \mathcal{P} \mathcal{T} \quad , \quad \mathcal{P} \Lambda \quad , \quad \mathcal{T} \Lambda \quad . \quad (33)$$

The condition (27) can be satisfied only for matrices of the first two kinds; these must have the form

$$\Lambda_m = \begin{pmatrix} m & 0 \\ 0 & \frac{1}{m} \end{pmatrix} \quad , \quad m \in \mathbb{Z} \quad . \quad (34)$$

This set of matrices (34) comprises the group eG_{lat} defined in section III; it is obviously a semigroup with composition and unit element

$$\Lambda_m \Lambda_{m'} = \Lambda_{mm'} \quad , \quad \mathbb{1} = \Lambda_1 \quad . \quad (35)$$

This semigroup is isomorphic to the set (\mathbb{Z}, \cdot) of all integers with multiplication as composition and one as unit element. For $m \neq 1$, the inverse of a matrix Λ_m in the full Lorentz group $O(1, 1)$ is given by

$$\Lambda_m^{-1} = \Lambda_{1/m} = \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & m \end{pmatrix} \quad . \quad (36)$$

However, these inverses violate the condition (27) and hence cannot be elements of eG_{lat} ; this fact accounts for the semigroup structure of (34). Consequently, the only Lorentz transformation (34) which preserves the lattice (25) such that its inverse preserves the lattice as well is the unit matrix $\mathbb{1}$; as a consequence, the group $G_{\text{lat}} = \{\mathbb{1}\}$ is trivial. Then (11) takes the form

$$N(\Gamma) \simeq \mathbb{R}^2 \quad , \quad (37a)$$

$$eN(\Gamma) \simeq \mathbb{R}^2 \odot (\mathbb{Z}, \cdot) \quad . \quad (37b)$$

In order to obtain the isometry group $I(C_r^2)$, and its semigroup extension $eI(C_r^2)$, we must divide out the lattice translations (26). Then, in lightlike coordinates (24),

$$I(C_r^2) \simeq \left\{ \begin{pmatrix} t^+ \\ t^- \end{pmatrix} \left| t^+ \in [0, 1) , t^- \in \mathbb{R} \right. \right\} \quad , \quad (38a)$$

$$eI(C_r^2) \simeq \left\{ \begin{pmatrix} t^+ \\ t^- \end{pmatrix} \left| t^+ \in [0, 1) , t^- \in \mathbb{R} \right. \right\} \odot (\mathbb{Z}, \cdot) \quad . \quad (38b)$$

(38a) says that the isometry group of the cylinder C_r^2 contains translations only; (38b) expresses the fact that the semigroup extension of the isometry group of the cylinder is given by the discrete Lorentz transformations (34). A faithful matrix representation of $eI(C_r^2)$ is obtained from (22) using lightlike coordinates (24),

$$(t, \Lambda_m) = \left(\begin{pmatrix} \Lambda_m & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m & 0 & t^+ \\ 0 & \frac{1}{m} & t^- \\ 0 & 0 & 1 \end{pmatrix} \right) \quad . \quad (39)$$

(39) acts on lightlike coordinates according to

$$(t, \Lambda_m) \begin{pmatrix} x^+ \\ x^- \end{pmatrix} = \begin{pmatrix} m x^+ + t^+ \\ \frac{1}{m} x^- + t^- \end{pmatrix} \quad . \quad (40)$$

We see that these transformations are volume-preserving, since translations and Lorentz transformations are so, and the transformations Λ_m are still *locally injective*; in particular, they are injective on the tangent spaces. Furthermore, it is obvious that Λ_m is injective on each of the strips $[0, \frac{r}{m}) \times \mathbb{R}$, $[\frac{r}{m}, \frac{2r}{m}) \times \mathbb{R}$, \dots , separately, but since each strip is mapped by Λ_m onto the whole interval $[0, 1)$, the inverse image $\Lambda_m^{-1}(x^+, x^-)$ of any point $(x^+, x^-) \in C_r^2$ contains m elements. Yet each of the Λ_m is perfectly metric-preserving,

$$\Lambda_m^T h \Lambda_m = h \quad , \quad (41)$$

and injective on tangent spaces, as mentioned above. These transformations therefore should qualify as isometries in an "extended" sense. What is more, even though these transformations are not globally injective, they still preserve the cardinality of the cylinder C_r^2 , since each of the strips $[\frac{r}{m}, \frac{2r}{m})$, etc., has the same cardinality as the total interval $[0, 1)$.

On the covering space \mathbb{R}_1^2 , transformations Λ_m are indeed one-to-one, and any two reference frames *on the covering* \mathbb{R}_1^2 which are related by such a transformation must be regarded as physically equivalent. This leads to an important question: Should reference frames on the compactification C_r^2 be regarded as equivalent if they are related by a semigroup transformation Λ_m (34)? Let us examine the consequences of such an assumption for a scalar field:

V. CLASSICAL SCALAR FIELDS LIVING ON C_r^2

Let us first clarify what we mean by a real scalar field on the compactification, by comparing it with the definition of an $O(1,1)$ -scalar ϕ on the covering space \mathbb{R}_1^2 : In \mathbb{R}_1^2 we have a set of equivalent frames which are mutually related by Lorentz transformations Λ . In each of these frames, an observer defines a single-valued field with respect to his coordinate chart by assigning $\mathbb{R}_1^2 \ni X \mapsto \phi(X) \in \mathbb{R}$ a single number to each point X . Consider any two equivalent frames with coordinates X and $X' = \Lambda X$, and call the respective field values $\phi(X)$ and $\phi'(X')$. If these field assignments are related by

$$\phi'(X') = \phi(\Lambda^{-1}X') \quad , \quad (42)$$

for all $X' \in \mathbb{R}_1^2$, and then for all pairs of equivalent frames, we refer to the collection of assignments ϕ, ϕ', \dots as an $O(1,1)$ scalar field.

If we now try to carry over this scenario to the compactified space C_r^2 we see that we run into certain troubles: The set $O(1,1)$ is reduced to a discrete set (34) which moreover is now only a semigroup, so that we cannot operate definition (42) which involves inverses of Λ . However, (42) may be rewritten in the form

$$\phi'(\Lambda X) = \phi(X) \quad , \quad (43)$$

for all $X \in C_r^2$, and $\Lambda = \Lambda_m$ now. Using (40) this can be written explicitly as

$$\phi' \left(m x^+, \frac{1}{m} x^- \right) = \phi(x^+, x^-) \quad . \quad (44)$$

Thus, if $\phi(X)$ is a single-valued field in frame/chart X , **and** if we adopt the assumption that frames on C_r^2 related by transformations (34) are physically equivalent, then frame/chart X' must see a **single-valued** field $\phi'(X')$ obeying (44). So, if x^+ ranges through $[0, \frac{r}{m})$, then x'^+ covers $[0, r)$ once; if x^+ ranges through $[\frac{r}{m}, \frac{2r}{m})$, then x'^+ already has covered $[0, r)$ twice, and so on. Thus, if ϕ' is supposed to be single-valued, then ϕ must be **periodic** with period r/m on the interval $[0, r)$ in the first place. But then this argument must hold for arbitrary values of m . Barring pathological cases and focusing on reasonably smooth fields ϕ we conclude that ϕ must be constant on the whole interval $[0, r)$! As a consequence of (44), each of the equivalent observers reaches the same conclusion for his field ϕ' . The statement that the scalar field ϕ is independent of the lightlike coordinate x^+ is therefore "covariant" with respect to the set of Λ_m -related observers on C_r^2 .

It is interesting to see that this fact makes the field ϕ automatically a solution to a massless field equation if ϕ was assumed to be massless in the first place: To see this, rewrite the d'Alembert operator $\square = -h^{ab}\partial_a\partial_b$ in lightlike coordinates using (28),

$$\square = 2\partial_+\partial_- = 2\partial_-\partial_+ \quad . \quad (45)$$

The equation of motion for a massless scalar field is $\square\phi = 0$, which, on account of (45), is seen to be satisfied automatically by all scalars which obey the consistency condition $\partial_+\phi = 0$ as discussed above. On the other hand, a massive scalar is inconsistent with the peculiar geometry on C_r^2 as manifested in the semigroup structure of the isometry group, since the massive Klein-Gordon equation $(\square + \mu_b^2)\phi = 0$ leads necessarily to $\mu_b^2\phi = 0$ for all scalars ϕ compatible with the semigroup structure on C_r^2 . Let us repeat this important point:

Any single-valued field ϕ on C_r^2 which transforms as a scalar with respect to the extended isometry group $eI(C_r^2)$ is necessarily a solution to a massless Klein-Gordon equation.

We can arrive at the same conclusion, and learn more, if we study plane-wave solutions to the massive Klein-Gordon equation:

$$\phi(x^0, x^1) \sim \cos(kx^1 - \omega_k x^0 + \delta) \quad , \quad \omega_k = \sqrt{k^2 + \mu_b^2} \quad . \quad (46)$$

Here we have started by assuming the general case, so that the dispersion law on the right-hand side of (46) is again that of a massive field, but we shall arrive at the conclusion for masslessness as above presently. Expressed in lightlike coordinates eq. (46) becomes

$$\phi(x^+, x^-) \sim \cos \left\{ \frac{1}{\sqrt{2}}(k - \omega_k)x^+ - \frac{1}{\sqrt{2}}(k + \omega_k)x^- + \delta \right\} \quad . \quad (47)$$

This quantity must be independent of x^+ , which is equivalent to saying that

$$\mu_b = 0 \quad , \quad k = |k| > 0 \quad . \quad (48)$$

In other words, only the *right-propagating* modes are admissible, while left-propagating ones, $k < 0$, are inconsistent with the $eI(C_r^2)$ -geometry on the compactification. The admissible modes, up to constant phase shifts, and expressed in $x^0 x^1$ -coordinates, then look like

$$\cos k(x^1 - x^0) = \Re f_k(x) \quad , \quad f_k(x) = e^{ikx^1 - ikx^0} = f_k(x^1 - x^0) \quad , \quad (49a)$$

so that any field ϕ may be decomposed into

$$\phi(x^1 - x^0) = \int_0^\infty \frac{dk}{4\pi k} \sqrt{\hbar_C} \left\{ a_k f_k(x) + a_k^\dagger f_k^*(x) \right\} \quad , \quad x = (x^0, x^1) \quad , \quad (49b)$$

where we have introduced the $SO(1,1)$ -invariant integration measure $dk/4\pi k$. At this point, \hbar_C is just a real numerical factor with dimension of an action which, in the quantum theory, may be identified with the Planck quantum of action on the cylinder C_r^2 .

From (49a) we immediately see that parity transformation or time-reversal do not map admissible solutions into admissible solutions; therefore, both of these symmetries are broken. However, a combined parity-time-reversal is admissible, as it maps a mode (49a) into itself. This is clearly consistent with the findings in (33, 34), where the combined transformation \mathcal{PT} was seen to be an isometry of the metric, but \mathcal{T} or \mathcal{P} separately were not.

Now, what about the dependence of ϕ on the coordinate x^- ? We can rewrite (44),

$$\phi'(x'^-) = \phi(m x'^-) \quad , \quad x'^- \in \mathbb{R} \quad . \quad (50)$$

Thus, an observer using chart $X' = \Lambda_m X$ sees a "contracted" version of the field, since, on an interval $[0, \frac{L}{m})$, x'^- has already covered the same information with respect to ϕ' as x^- in the interval $[0, L)$ with respect to ϕ . More precisely, the Fourier transforms $\phi(k)$ and $\phi'(k')$ of the fields $\tilde{\phi}(X)$ and $\tilde{\phi}'(X')$ are related by

$$\tilde{\phi}'(k') = \frac{1}{m} \tilde{\phi}\left(\frac{k'}{m}\right) \quad , \quad (51)$$

hence the frequency spectrum of the scalar field is "blue-contracted", thus "more energetical", for the X' -observer. Suppose that the field ϕ is such that it tends to zero at infinity. Then in the "infinite-momentum" limit $m \rightarrow \infty$ we see a field with spatial dependence

$$\phi'(x'^-) = \begin{cases} 0 & , \quad x'^- \neq 0 \\ \phi(0) & , \quad x'^- = 0 \end{cases} \quad (52)$$

This result is noteworthy: We have started out with a field ϕ which was supposed to transform under the semigroup transformations as a scalar. This transformation behaviour has led us to conclude that the field must be right-propagating, and must not depend on the coordinate x^+ . We have derived this independence by assuming only that 1.) the lightlike direction x^+ is compactified, with no condition on the length r of the compactified interval; and 2.) that the semigroup transformations (34) continued to make sense as maps between physically equivalent reference frames, which allowed us to define fields with scalar transformation behaviour under the semigroup (34). On the other hand, we must keep in mind that the transformations Λ_m no longer comprise a group, but only a semigroup with the property (35) that the succession of two such transformations can ever go only into **one** direction, namely towards $m \rightarrow \infty$. In our view this points out the infinite-momentum frame $m \rightarrow \infty$ as something preferred! The preference is manifested in the fact that the set of Lorentz transformations (34) now has an *inherent order*, given by

$$\Lambda_m \prec \Lambda_{m+1} \prec \cdots \quad , \quad (53)$$

which follows the order $m < m+1 < \cdots$ of the indices m . This order reflects the fact that a transformation $Y = \Lambda_m X$ between two different frames cannot be undone (after all, it is not globally injective). This is in stark contrast to the usual case, for example, the relation between two different frames on the covering space \mathbb{R}_1^2 , where both frames can be reached from each other by an appropriate Lorentz transformation. We see that the ordering (53) clearly points out the "infinite-momentum" frame at $m \rightarrow \infty$ as a *preferred frame*. This preferred frame is an immediate consequence of admitting the semigroup transformations (34) to be part of our physical reasoning on the spaces C_r^2 . Some consequences of these preferred frames for Kaluza-Klein theories will be presented in section VII below. First, however, we must study the transformation behaviour of vectors and covectors under the semigroup transformations (34).

VI. VECTORS AND COVECTORS ON C_r^2

We can extend the reasoning leading to the scalar transformation law (44) to vector and covector fields on C_r^2 . Consider a vector field $V = V^+ \partial_+ + V^- \partial_-$ on C_r^2 ; then V must transform under Λ_m -transformations as $V'(\Lambda_m X) = \Lambda_m V(X)$, or

$$\begin{bmatrix} V'^+(mx^+, \frac{1}{m}x^-) \\ V'^-(mx^+, \frac{1}{m}x^-) \end{bmatrix} = \begin{bmatrix} m V^+(x^+, x^-) \\ \frac{1}{m} V^-(x^+, x^-) \end{bmatrix} . \quad (54)$$

As in the case of scalar fields, single-valuedness of the component fields requires that the fields cannot depend on x^+ .

Now we consider 1-forms $\omega = \omega_+ dx^+ + \omega_- dx^-$, denoting components by row vectors (ω_+, ω_-) . Their transformation behaviour is obtained from

$$\omega = \omega_+ dx^+ + \omega_- dx^- = \omega'_+ dx'^+ + \omega'_- dx'^- , \quad (55)$$

such that

$$\left(\omega'_+(\Lambda_m X), \omega'_-(\Lambda_m X) \right) = \left(\omega_+(X), \omega_-(X) \right) \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & m \end{pmatrix} = (\omega_+, \omega_-) \Lambda_m^{-1} . \quad (56)$$

Again, we conclude that the fields cannot depend on x^+ . This latter feature is clearly general, and applies to tensor fields of arbitrary type $\binom{m}{n}$, since it arises from the transformation of the arguments X of the tensor fields, rather than the transformation of the tensor components.

VII. CONSEQUENCES FOR SIX-DIMENSIONAL KALUZA-KLEIN THEORIES

So far we have studied the consequences of admitting semigroup transformations to connect different physical frames on the cylindrical spacetimes C_r^2 . Now we want to examine these transformations within a greater framework: We envisage the case of a product spacetime $M \times C_r^2$, where the external factor M is a standard 3+1-dimensional Minkowski spacetime with metric η , while the internal factor is comprised by our cylindrical space C_r^2 with metric $h_{ab} = \text{diag}(-1, 1)$. We use coordinates $(x^\mu, x^a) = x^A$ on the product manifold such that $x^\mu, \mu = 0, \dots, 3$, and $x^a, a = 4, 5$, are coordinates on the external Minkowski space and the internal cylinder, respectively. In the absence of gauge fields, the signature of the 6-dimensional metric is $(-1, 1, 1, 1, -1, 1)$. When off-diagonal metric components are present, we denote the metric as

$$\hat{g} = \left(\begin{array}{c|c} g_{\mu\nu} & A_{\mu a} \\ \hline A_{a\nu} & h_{ab} \end{array} \right) , \quad (57)$$

or explicitly,

$$\hat{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu + A_{a\mu} (dx^a \otimes dx^\mu + dx^\mu \otimes dx^a) + h_{ab} dx^a \otimes dx^b . \quad (58)$$

Here, a, b, \dots run over indices $+$ and $-$ on the internal dimensions; while Greek indices range over the external dimensions $0, \dots, 3$ on M . The submetric $g_{\mu\nu}(x^\rho, x^a)$ of the external spacetime may turn out to depend on the internal coordinates x^a , so all we may demand is that on the 3+1-dimensional submanifold we are aware of, i.e. for $x^a = 0$, we have $g_{\mu\nu}(x^\rho, 0) = \eta_{\mu\nu}$.

Let us now apply a semigroup transformation (34) to (58),

$$\hat{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu + A_{a\mu} (\Lambda_m^{-1})^a{}_b (dy^b \otimes dx^\mu + dx^\mu \otimes dy^b) + h_{ab} dy^a \otimes dy^b , \quad (59)$$

where $y^a = (\Lambda_m)^a{}_b x^b$, and the internal part of the metric remains invariant, since Λ_m preserves the metric h_{ab} . Comparison with (56) shows that the off-diagonal elements $A_{a\mu}$ transform like covectors on the internal manifold; we therefore conclude that they must be independent of the x^+ -coordinate on C^2 . Hence,

$$\begin{aligned} A'_{+\mu}(y^-) &= \frac{1}{m} A_{+\mu}(my^-) , \\ A'_{-\mu}(y^-) &= m A_{-\mu}(my^-) , \end{aligned} \quad (60)$$

where, for sake of simplicity, we have temporarily suppressed the dependence of the gauge potentials $A_{a\mu}$ on the spacetime coordinates x^μ . We now assume that the off-diagonal elements satisfy

$$\lim_{x^- \rightarrow \pm\infty} A_{a\mu}(x^-) = 0 \quad . \quad (61)$$

Then the same arguments given in (50, 52) lead to the conclusion that, in the "infinite-momentum" limit $m \rightarrow \infty$, the $A_{+\mu}$ -fields vanish, while the $A_{-\mu}$ -fields appear localized around the point $x^- = 0$, but acquire an infinite amplitude. This latter feature is less catastrophic than it might appear, though: There is no inherent length scale on the internal manifold which must be preserved, and so nothing can stop us from applying a conformal rescaling

$$\begin{pmatrix} y^+ \\ y^- \end{pmatrix} \rightarrow \begin{pmatrix} z^+ \\ z^- \end{pmatrix} = \begin{pmatrix} y^+ \\ e^\beta y^- \end{pmatrix} \quad , \quad (62)$$

which is to accompany the limit $m \rightarrow \infty$ in such a way as to compensate for the factor of m in front of $A_{-\mu}$. Explicitly, the metric (59) in the coordinates (62) becomes

$$\begin{aligned} \hat{g} = & \eta_{\mu\nu} dx^\mu \otimes dx^\nu + \\ & + A_{+\mu} \frac{1}{m} (dz^+ \otimes dx^\mu + dx^\mu \otimes dz^+) + \\ & + e^{-\beta} A_{-\mu} m (dz^- \otimes dx^\mu + dx^\mu \otimes dz^-) + \\ & + e^{-\beta} h_{ab} dz^a \otimes dz^b \quad . \end{aligned} \quad (63)$$

Hence, if we perform the limits $\beta \rightarrow \infty$ and $m \rightarrow \infty$ simultaneously by setting $e^\beta = m$, then

$$\hat{g} \xrightarrow{m \rightarrow \infty} \eta_{\mu\nu} dx^\mu \otimes dx^\nu + A_{-\mu}(x^\nu, z^-) (dz^- \otimes dx^\mu + dx^\mu \otimes dz^-) + 0 \cdot dz^- \otimes dz^- \quad . \quad (64)$$

Eq. (64) is a very interesting result: As seen from the preferred, infinite-momentum, frame, the field content of the original metric (57) is that of a Kaluza-Klein theory defined on a flat $(3+1)$ -dimensional external Minkowski space times a one-dimensional factor (the z^- -direction) along which the metric is zero, while the z^+ -direction has completely vanished out of sight. In a more technical language, we obtain an effective 5-dimensional *non-compactified* [15] Kaluza-Klein theory. The associated metric for $z^- = 0$ is

$$\hat{g} = \left(\begin{array}{c|c} \eta_{\mu\nu} & A_{\mu-} \\ \hline A_{-\nu} & 0 \end{array} \right) \quad . \quad (65)$$

In contrast to previous Kaluza-Klein theories, the metric (65) is *degenerate* in the absence of fields; furthermore, it may be expected that the signature of the overall metric \hat{g} , now including gauge potentials, will depend on the particular field configuration represented by the $A_{-\mu}$: The characteristic equation of the metric (57) is

$$(\lambda - 1)^2 \left\{ \lambda^3 - \lambda (\mathbf{A}^2 + A_0^2 + 1) - \mathbf{A}^2 + A_0^2 \right\} = 0 \quad , \quad (66)$$

where we have abbreviated $A_\mu \equiv A_{-\mu}$. Two eigenvalues are always equal to one. If the vector potential satisfies $A_\mu A^\mu = 0$ then a zero eigenvalue exists, and the metric is degenerate even in the presence of fields A_μ . We have checked numerically that both cases $\mathbf{A}^2 = 0$ and $A_0^2 = 0$ can produce negative, vanishing and positive eigenvalues.

The emergence of a degenerate metric clearly goes beyond the framework of standard Kaluza-Klein theories, where the non-degeneracy of the overall metric is usually taken for granted. However, we are inclined to take the result (64) serious if it can be shown to produce meaningful physics. To this end we have to contemplate how equations of motion for the fields $A_\mu \equiv A_{-\mu}$ can be obtained. The point here is that the usual technique of obtaining dynamical equations for the fields from the Einstein equations for the total metric \hat{g} no longer works: On account of the fact that the inverse \hat{g}^{-1} does not exist in the field-free case, a Levi-Civita connection on the total manifold is not well-defined, since the associated connection coefficients (Christoffel symbols) involve the inverse of the metric. It might be possible to produce equations of motions for the fields A_μ by going back to the original field multiplet $A_{a\mu}$, before taking the limit of the infinite-momentum frame on the internal space; computing the Ricci tensor, and then trying to obtain a meaningful limit $m \rightarrow \infty$ for the theory even though the metric becomes degenerate in this limit. This possibility will be explored elsewhere.

VIII. SUMMARY

The concept of the extended normalizer of a group of isometries leads to the possibility of semigroup extensions of isometry groups of compactified spaces. For flat covering spaces which are compactified over lattices, semigroup extensions become possible when the lattice contains lightlike vectors. The simplest example is provided by the family of two-dimensional cylindrical spacetimes with Lorentzian signature compactified along a lightlike direction; the members of this family are all isometric to each other. The semigroup elements acting on these cylinders can be given a natural ordering, which in turn suggests the existence of preferred coordinate frames, the latter consisting of infinite-momentum frames related to the canonical chart by extreme Lorentz transformations. Fields as viewed from the preferred frame acquire extreme values; in particular, some of the off-diagonal components of the total metric, regarded as gauge potential for a field theory on the external Minkowski factor, may vanish, leaving a field multiplet which is reduced in size and which no longer depends on one of the coordinates on the internal space. The infinite amplitude exhibited by the surviving fields can be removed by conformal rescaling of one of the lightlike directions. The effective theory so obtained is a Kaluza-Klein theory which is reduced by one dimension, and has a smaller field content, but which is defined on a space with degenerate metric. The latter feature is responsible for the fact that dynamical equations for the fields A_μ cannot be obtained from the Ricci tensor of the overall metric.

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